Lines with the butterfly property

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Abstract. *In this paper it is explored which lines have the butterfly property with respect to quadrangles (inscribed into a given conic curve).*

Key words: *butterfly property, line, conic, inscribed quadrangle, midpoint*

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Let $ABCD$ be a plane quadrangle, w a line intersecting all sides and diagonals of $ABCD$ (considered as lines), and S a point on w. Let H, K, U, V, X, and Y denote intersections of w with lines AB, CD, AC, BD, AD, and BC, respectively. We consider the statements

$\mathbb{B}(w, ABCD)$**:** If the midpoints of any two of the following segments HK, UV, and XY coincide, then they all coincide.

$\mathbb{B}(w, S, ABCD)$**:** If S is the midpoint of any of the following segments HK, UV, and XY, then it is the midpoint of them all.

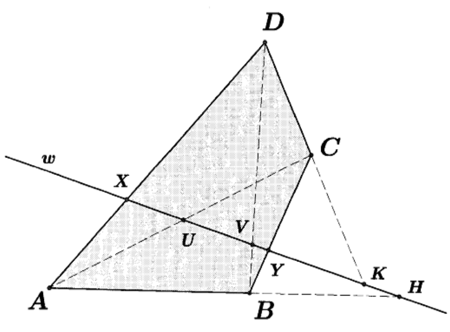

Figure 1. *Quadrangle $ABCD$ and six points of intersection of its sides with line w*

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The first statement $\mathbb{B}(w,\, ABCD)$ is not very interesting because we have the following result (see *Figure 1*).

Theorem 1. *The statement $\mathbb{B}(w,\, ABCD)$ is true for every line w and every quadrangle $ABCD$.*

Proof. Without loss of generality, we can assume that A, B, C, D are points in the Gauss complex plane with affixes 0 (zero), 1 (one), c, and d and that the line w has the equation $z + t\,\bar{z} = s$, where t is a unimodular complex number and $S = (s)$ is the point symmetric to the origin with respect to the line w.

This unusual equation for a line in the complex plane is explained on page 76 of the reference [5] and could be seen as follows. Without loss of generality, one can assume that no vertex of $ABCD$ belongs to w. Then one can consider w as the perpendicular bisector of segment AS which leads to the equation in the given form.

The points of intersection have the following affixes $h = \frac{s}{1+t}$, $k = \frac{(t\,\bar{d}-s)\,c+(s-\bar{c}\,t)\,d}{d-c+t\,(\bar{d}-\bar{c})}$, $u = \frac{c\,s}{c+\bar{c}\,t}$, $v = \frac{(s-t)\,d+t\,\bar{d}-s}{d-1+t\,(\bar{d}-1)}$, $x = \frac{s\,d}{d+t\,\bar{d}}$, and $y = \frac{(c-1)\,s+(\bar{c}-c)\,t}{(\bar{c}-1)\,t+c-1}$. Now $h_2 = \frac{1}{2}\,(h+k)$, $u_2 = \frac{1}{2}\,(u+v)$, and $x_2 = \frac{1}{2}\,(x+y)$ are the affixes of the midpoints H_2, U_2, and X_2 of HK, UV, and XY respectively. Using as denominators for $h_2 - u_2$ and $h_2 - x_2$ just the products of the denominators in the given descriptions of h, k, u, v, x, and y, one finds that fractions describing $h_2 - u_2$ and $h_2 - x_2$ have the same numerator (possibly up to the sign). From this the conclusion of the theorem follows immediately. Indeed, if H_2 and U_2 coincide, then the numerator of $h_2 - u_2$ vanishes and so does the numerator of $h_2 - x_2$ implying finally $H_2 = X_2$. $\square$

Remark 1. *The hypothesis that the line w intersects all sides and diagonals is essential in* Theorem 1. *In the case of an isosceles trapezium $ABCD$ and $w \| AB \| CD$ the midpoints of UV and XY coincide while the points H and K do not exist.*

Our goal now is to prove the following three theorems.

Theorem 2. *For every parabola k and every point S there is a unique line w such that $\mathbb{B}(w,\, S,\, ABCD)$ is true for every quadrangle $ABCD$ inscribed into k.*

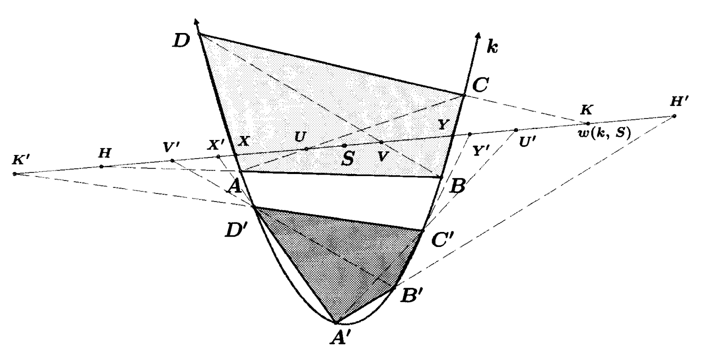

Figure 2. *Parabola k and point S with line $w(k,\, S)$ and two inscribed quadrangles having the butterfly property with respect to this line and the point*

Theorem 3. *Let O be the centre of either an ellipse or hyperbola k. For every line w through O the statement $\mathbb{B}(w,\ O,\ ABCD)$ is true for every quadrangle $ABCD$ inscribed into k.*

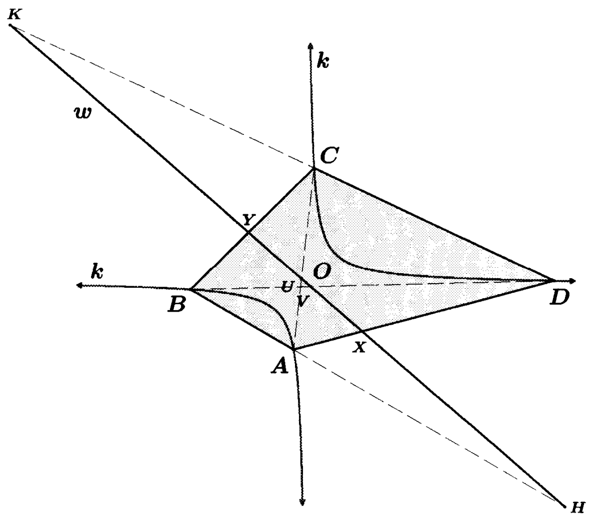

Figure 3. *Hyperbola k and line w through the centre O with an inscribed quadrangle such that $\mathbb{B}(w,\ O,\ ABCD)$ holds*

Theorem 4. *If k is either an ellipse or a hyperbola with the centre O, then for every point S different from O there is a unique line w such that $\mathbb{B}(w,\ S,\ ABCD)$ is true for every quadrangle $ABCD$ inscribed into k.*

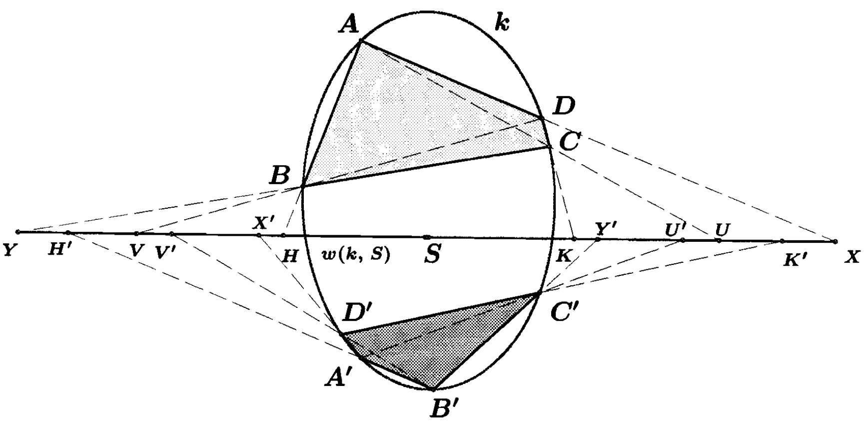

Figure 4. *Ellipse k, point S and line $w = w(k,\ S)$ through S with inscribed quadrangles such that $\mathbb{B}(w,\ S,\ ABCD)$ and $\mathbb{B}(w,\ S,\ A'B'C'D')$ hold*

Proof. Before proving these theorems we shall recall some facts from the analytic geometry of conics. It is well-known that if we take a focus of conic k as the pole (the origin) and the main axis (the line of symmetry through the focus) μ as the polar axis of a polar coordinate system, then k has the equation $\varrho = p/(1 + \varepsilon\,\cos\vartheta)$, where ϱ is the polar radius, ϑ is the polar angle, and p and ε are nonnegative real numbers. Hence, in the associated rectangular coordinate system points A, B, C, and D have coordinates $(p\,\cos\vartheta/(1+\varepsilon\,\cos\vartheta), p\,\sin\vartheta/(1+\varepsilon\,\cos\vartheta))$, where ϑ is α, β, γ, and δ. We could continue using trigonometric functions but it is easier at this point to employ the universal trigonometric substitution to write

$$\cos\alpha = \frac{1-a^2}{1+a^2}, \qquad \sin\alpha = \frac{2\,a}{1+a^2},$$

and similarly for the remaining three points (and their corresponding letters). We conclude that points A, B, C, and D have coordinates

$$\left(\frac{p\,(1-t^2)}{\varepsilon\,(1-t^2)+t^2+1}, \frac{2\,p\,t}{\varepsilon\,(1-t^2)+t^2+1}\right)$$

for t equal to a, b, c, and d.

Let us assume that line w has the equation $f\,x + g\,y + h = 0$ and that point S has coordinates $(m,\,n)$. Since point S belongs to line w it follows that $h = -f\,m - g\,n$. Line AB has the equation

$$(a\,b\,(\varepsilon-1)+\varepsilon+1)\,x + (a+b)\,y - p\,(a\,b+1) = 0.$$

The other lines CD, AC, BD, AD, and BC have analogous equations. The point of intersection H of lines w and AB has the coordinates

$$\left[\frac{g\,p\,(a\,b+1)+h\,(a+b)}{g\,(\varepsilon-1)\,a\,b - f\,(a+b)+g\,(\varepsilon+1)}, \frac{-h\,((\varepsilon-1)\,a\,b+\varepsilon+1) - f\,p\,(a\,b+1)}{g\,(\varepsilon-1)\,a\,b - f\,(a+b)+g\,(\varepsilon+1)}\right].$$

Notice that the denominators of the above fractions do not vanish since the considered point of intersection exists by hypothesis. The other points of intersection K, U, V, X, and Y have similar coordinates.

Let $H_2(h_2,\,k_2)$, $U_2(u_2,\,v_2)$, and $X_2(x_2,\,y_2)$ be the midpoints of the segments HK, UV, and XY. Then $h_2 - m = \frac{g\,M_H}{2\,N_H\,P_H}$ and $k_2 - n = -\frac{f\,M_H}{2\,N_H\,P_H}$, where

$$P_H = (c+d)f + cd(1-\varepsilon)g - (1+\varepsilon)g,\;\; N_H = (a+b)f + ab(1-\varepsilon)g - (1+\varepsilon)g,$$

$$M_H = m\,Q_H + n\,R_H + p\,S_H, \quad Q_H = -\mathcal{Z}\,\varepsilon^2 + (\mathcal{P}f + 2\mathcal{U}g)\varepsilon + \mathcal{D}g - \mathcal{R},$$

$$R_H = \mathcal{S}f + (\mathcal{R} - \mathcal{P}\varepsilon)g, \quad S_H = \mathcal{Z} - \mathcal{P}f - \mathcal{U}g,$$

with $\mathcal{Z} = 2\,(a\,b+1)(c\,d+1)$, $\mathcal{D} = 2\,(a\,b-1)(c\,d-1)$, $\mathcal{S} = 2\,(a+b)(c+d)$, $\mathcal{U} = 2\,(a\,b\,c\,d-1)$, $\mathcal{P} = a\,b\,c + a\,b\,d + a\,c\,d + b\,c\,d + a + b + c + d$, and $\mathcal{R} = a\,b\,c + a\,b\,d + a\,c\,d + b\,c\,d - a - b - c - d$.

Notice that

$$M_H - M_U = 2\,(d-a)\,(b-c)(n\,f + [m\,(\varepsilon^2-1) - p\,\varepsilon]\,g)$$

and

$$M_H - M_X = 2\,(d-b)\,(a-c)(n\,f + [m\,(\varepsilon^2-1) - p\,\varepsilon]\,g).$$

Without loss of generality, we now assume $H_2 = S$, i. e., that $M_H = 0$. Then we have to look for conditions on line w implying $U_2 = X_2 = S$, i. e., $M_U = M_X = 0$.

When k is a parabola, then $\varepsilon = 1$ so that we distinguish two possibilities: (a) $n = 0$ and (b) $n \neq 0$.

In the first case, point S belongs to axis μ of k and it follows

$$M_U = M_X = 0 \Leftrightarrow g = 0.$$

But $g = 0$ means that w is the line perpendicular to the axis of the parabola passing through point S.

In the second case, point S is not on axis μ of k and points U_2, and X_2 coincide with point S if and only if $f = \frac{p\,g}{n}$ (i. e., if and only if w has the equation $p\,x + n\,y = m\,p + n^2$). This proves *Theorem 2*.

When k is either an ellipse or a hyperbola, then $\varepsilon \neq 1$ and its centre is at point $O(\frac{p\,\varepsilon}{\varepsilon^2-1},\,0)$. Now we distinguish four cases: (i) $(m,\,n) = (\frac{p\,\varepsilon}{\varepsilon^2-1},\,0)$ (i. e., $S = O$), (ii) $n = 0$ and $m \neq \frac{p\,\varepsilon}{\varepsilon^2-1}$, (iii) $n \neq 0$ and $m = \frac{p\,\varepsilon}{\varepsilon^2-1}$ and (iv) $n \neq 0$ and $m \neq \frac{p\,\varepsilon}{\varepsilon^2-1}$.

In case (i), we have $M_U = M_X = 0$ so that $\mathbb{B}(w,\,O,\,ABCD)$ is true for every line w which goes through the center O of either an ellipse or a hyperbola k and for every quadrangle $ABCD$ inscribed into it. This proves *Theorem 3*.

In case (ii), point S is on the principal axis μ of k and points H_2, U_2, and X_2 coincide with point S if and only if $g = 0$ (i. e., if and only if w is perpendicular to μ at point S).

In case (iii), point S is on the secondary axis ν of k and points H_2, U_2, and X_2 coincide with point S if and only if $f = 0$ (i.e., if and only if w is the perpendicular to ν at point S).

Finally, in case (iv), point S is not on either axis of k and points H_2, U_2, and X_2 coincide with point S if and only if

$$f = \frac{(p\,\varepsilon - m\,(\varepsilon^2-1))\,g}{n}$$

(with $g \neq 0$), i.e., if and only if w has the equation

$$(p\,\varepsilon - m\,(\varepsilon^2-1))\,x + n\,y = m\,(p\,\varepsilon - m\,(\varepsilon^2-1)) + n^2.$$

This proves *Theorem 4*.

Line w from *Theorems 2* and *4* is denoted also as $w(k,\,S)$. The above proof establishes also the following corollary which is the main result in [3] and [2].

Corollary 1. *Let k be a conic and let S be a point different from the centre of k (if the centre exists). Line $w(k,\,S)$ is perpendicular to axis z of k if and only if S lies on z.*

Our second corollary shows that the main result in [10] is also covered by the above theorems.

Corollary 2. *Let k be a conic and let ℓ be a line in the same plane. If S is the point of intersection of ℓ with the diameter of k conjugate to ℓ and S is different from the centre of k (when the centre exists), then $w(k,\,S) = \ell$.*

Proof. We know that line $w(k,\, S)$ has the equation

$$(p\,\varepsilon - m\,(\varepsilon^2-1))\,x + n\,y - m\,(p\,\varepsilon - m\,(\varepsilon^2-1)) - n^2 = 0$$

where $(m,\, n)$ are coordinates of S. In order to find these coordinates, let us assume that line ℓ has the equation $f\,x + g\,y + h = 0$. In the rectangular coordinate system k has the equation $(\varepsilon^2-1)\,x^2 - y^2 - 2\,\varepsilon\,p\,x + p^2 = 0$. When we compute the midpoint of the points of intersections of k and ℓ and eliminate parameter h we obtain the equation $(\varepsilon^2-1)\,g\,x + f\,y - \varepsilon\,p\,g = 0$ of the diameter of k conjugate to the given line ℓ. It intersects line ℓ at the point

$$S\left(-\frac{\varepsilon\,p\,g^2 + f\,h}{f^2 + g^2\,(1-\varepsilon^2)},\, \frac{g\,(\varepsilon\,p\,f + h\,(\varepsilon^2-1))}{f^2 + g^2\,(1-\varepsilon^2)}\right).$$

By substituting the coordinates of S for m and n on the left-hand side of the above equation of $w(k,\, S)$ we shall get

$$\frac{\big(\varepsilon\,p\,f + h\,(\varepsilon^2-1)\big)\,(f\,x + g\,y + h)}{f^2 + g^2\,(1-\varepsilon^2)}.$$

This clearly concludes the proof. □

The next result shows the connection of our theorems with the version of the original Butterfly Theorem from [7] and the Three-Winged Butterfly Problem from [8] for conics.

Theorem 5. *If S is the midpoint of chord PQ of conic k, then $w(k,\, S)$ is line PQ.*

Proof. From the proof of *Theorems 2–4* we know that line $w(k,\, S)$ has the equation $f\,x + g\,y = f\,m + g\,n$ where $(m,\, n)$ are coordinates of S and

$$f\,n + g\,(m\,(\varepsilon^2-1) - p\,\varepsilon) = 0. \tag{1}$$

We assume that P and Q have coordinates

$$\left(\frac{p\,(1-t^2)}{\varepsilon\,(1-t^2) + t^2 + 1},\, \frac{2\,p\,t}{\varepsilon\,(1-t^2) + t^2 + 1}\right)$$

for t equal to u and v. It follows that by substituting for m and n the coordinates of the midpoint of the segment PQ into (1) we obtain

$$\frac{p\,((\varepsilon-1)\;u\,v - \varepsilon - 1)(-\,(u+v)\;f + (u\,v\,(\varepsilon-1) + \varepsilon + 1)\,g)}{((\varepsilon-1)\,u^2 - \varepsilon - 1)((\varepsilon-1)\,v^2 - \varepsilon - 1)} = 0.$$

Since the equation of line PQ is $(u\,v\,(\varepsilon-1) + \varepsilon + 1)\,x + (u+v)\,y = p\,(u\,v + 1)$ it is obvious that $w(k,\, S) = PQ$. □

Remark 2. *Line $w(k,\, S)$ has the following simple construction. When k is a parabola with directrix d, then the perpendicular through S to d intersects k at point P and $w(k,\, S)$ is the parallel through S to the tangent at P to k. When k is an ellipsis or a hyperbola and S is different from the centre O of k, then line OS intersects k at point P (which could be imaginary) and $w(k,\, S)$ is the parallel through S to the tangent at P to k.*

Remark 3. *This paper (without* Corollary 2*) was written in August 2001. In the meantime, [10] has appeared which is similar in that for a given line w it searches for a point S on it such that $\mathbb{B}(w, S, ABCD)$ is true while our approach is to find a line w through a given point S such that $\mathbb{B}(w, S, ABCD)$ holds.*

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